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Computing $\mathbf{L_2}$-Gain of Finite-Horizon Systems With Boundary Conditions

Hisaya Fujioka

***Abstract*—A bisection algorithm is developed for computing the $\mathbf{L}_2$-gain of a finite-horizon system with boundary conditions. Upper and lower bounds of the gain are also derived for the initial step of the algorithm.**

***Index Terms*—$\mathbf{L}_2$-gain, boundary conditions, finite-horizon systems.**

I. Problem Formulation and Motivations

Consider a linear state-space system

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{1}$$

with constraints on the state

$$\Omega x(0) + \Upsilon x(1) = 0 \tag{2}$$

where A, B, C, and D are real matrices of compatible dimensions and Ω and Υ are square real matrices. The following two statements are equivalent [9].

i) Equations (1) and (2) has a unique solution $x = 0$ for $u = 0$.
ii) The matrix Ξ is nonsingular, where

$$\Xi := \Omega + \Upsilon e^{A}. \tag{3}$$

Supposing the nonsingularity of Ξ, we can define an operator $G\colon u \mapsto y$ on $\mathbf{L}_2[0, 1]$ by (1) and (2).

The purpose of this note is to develop an algorithm to compute the norm of G:

$$\|G\| := \sup_{u \in \mathbf{L}_2[0,1]} \frac{\|y\|_2}{\|u\|_2} \tag{4}$$

where $\|\cdot\|_2$ denotes the standard $\mathbf{L}_2$ norm

$$\|f\|_2 = \sqrt{\int_0^1 f^*(t) f(t)\, \mathrm{d}t}.$$

This problem is motivated as follows: The computation of $\|G\|$ with $\Upsilon = 0$ has been studied for robust control problems for delay systems (e.g., [11]) and sampled-data systems (e.g., [1], [7]). In particular, the algorithm in [3] includes no approximation such as gridding, and hence provides a reliable computational method. However, we encounter the case with nonzero Υ, for which there is no efficient algorithm at present, in applications including: i) the spatio-temporal frequency response gain of a class of infinite-dimensional systems [6], ii) the frequency response gain of sampled-data systems (see, e.g, Proposition 2 for computing γ_L in [10]), and iii) the worst case power ratio of periodic inputs/outputs (see Section IV).

In this note, we develop a bisection algorithm to compute $\|G\|$ by extending that in [3]. The formula is also improved with smaller-sized matrix exponentials even for the case of $\Upsilon = 0$. We also derive upper and lower bounds of $\|G\|$ to complete the algorithm.

II. Main Results

In this section, we provide a finite dimensional condition to check whether $\|G\| < \gamma$ or not for given $\gamma > 0$.

In the sequel we assume that $\gamma > \sigma_{\max}(D)$. Note that we can assume this without loss of generality since $\|G\| \geq \gamma$ automatically holds if $\sigma_{\max}(D) \geq \gamma$. Under the assumption, the following Hamiltonian matrices H and $H_{\min}$ are well-defined

$$H := \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C}$$
$$H_{\min} := \begin{bmatrix} -A_{\min}^* & -C_{\min}^* C_{\min} \\ 0 & A_{\min} \end{bmatrix} + \begin{bmatrix} -C_{\min}^* D \\ B_{\min} \end{bmatrix} \tilde{D}^{-1} \begin{bmatrix} B_{\min} \\ C_{\min}^* D \end{bmatrix}^*$$

where

$$\tilde{A} := \begin{bmatrix} -A^* & -C^*C \\ 0 & A \end{bmatrix} \tilde{B} := \begin{bmatrix} -C^*D \\ B \end{bmatrix} \tilde{C} := [\, B^* \quad D^*C \,]$$
$$\tilde{D} := \gamma^2 I - D^* D$$

and $(A_{\min}, B_{\min}, C_{\min}, D)$ is given as a minimal realization of the transfer function related to G

$$C(sI - A)^{-1}B + D. \tag{5}$$

Note that we cannot take (1) be minimal as a realization of (5) in general because of the boundary condition [9].

The following theorem is the main result of this note.

Theorem 1: Given G and $\gamma > \sigma_{\max}(D) > 0$. The following two statements are equivalent.

i) $\|G\| < \gamma$.
ii) $\rho(\Gamma) < \gamma^2$ where $\rho(\cdot)$ denotes the spectral radius, and the matrix Γ is defined as follows.

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Step 1: Fix $\theta \in (-\pi, \pi]$ such that

$$\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^{A}) \quad \mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^{H}).$$

Step 2: Define M and W_∞ by

$$\begin{aligned} M &:= M_r^* \begin{bmatrix} Q & (\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A})^* \\ \mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A} & 0 \end{bmatrix} M_r \\ W_\infty &:= \frac{1}{2} J \left(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{H}\right)^{-1} \left(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{H}\right) \\ &\quad - \frac{1}{2} \Lambda_r^* \begin{bmatrix} 0 & -(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{A}) \\ -(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{A})^* & 2Q \end{bmatrix} \Lambda_r \end{aligned}$$

where

$$\begin{aligned} M_r &:= \begin{bmatrix} \Xi^{-1}(\Omega \mathrm{e}^{-\mathrm{j}\theta} + \Upsilon) & 0 \\ 0 & I \end{bmatrix} \quad Q := \int_0^1 \mathrm{e}^{A^* t} C^* C \mathrm{e}^{At}\, \mathrm{d}t \\ \Lambda_r &:= \begin{bmatrix} I & 0 \\ 0 & (\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A})^{-1} \end{bmatrix} \quad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \end{aligned} \tag{6}$$

Step 3: If $H_{\min}$ has an eigenvalue on the imaginary axis

$$\eta := \max\{|\omega| : \omega \in \mathbb{R} \quad \mathrm{j}\omega \in \mathrm{eig}(H_{\min})\} \geq 0$$

is well-defined. Fix N as a nonnegative integer satisfying

$$|\omega_{N+1}| > \eta \quad |\omega_{N+2}| > \eta$$

where $\{\omega_i\}_{i=0}^\infty$ is defined by

$$\omega_i := 2\pi v_i + \theta \quad \{v_i\}_{i=0}^\infty := \{0, 1, -1, 2, -2, \ldots\}.$$

Then, Γ is defined by

$$\Gamma := \left(\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L^* \\ I \end{bmatrix} M \begin{bmatrix} L & I \end{bmatrix}\right) \begin{bmatrix} I & 0 \\ 0 & \gamma^2 (W_\infty - W_N) \end{bmatrix}$$

where

$$\begin{aligned} K &:= \mathrm{diag}(P_0^* P_0, \ldots, P_N^* P_N) \quad L := [S_0 \ \cdots \ S_N] \\ S_i &:= \begin{bmatrix} -(\mathrm{j}\omega_i I - A)^{-1} B \\ (\mathrm{j}\omega_i I - A)^{-*} C^* P_i \end{bmatrix} \quad P_i := C(\mathrm{j}\omega_i I - A)^{-1} B + D \end{aligned}$$

and W_N is defined by

$$\begin{aligned} W_N &:= -\sum_{i=0}^{N} \left(\begin{bmatrix} C^* C & (\mathrm{j}\omega_i I - A)^* \\ \mathrm{j}\omega_i I - A & 0 \end{bmatrix} + \tilde{B} \tilde{D}^{-1} \tilde{B}^*\right)^{-1} \\ &\quad + \sum_{i=0}^{N} \begin{bmatrix} I & 0 \\ 0 & (\mathrm{j}\omega_i I - A)^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & -C^* C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (\mathrm{j}\omega_i I - A)^{-1} \end{bmatrix}. \end{aligned}$$

If not the case, i.e., if $H_{\min}$ has no pure imaginary eigenvalue, Γ is defined by

$$\Gamma := \gamma^2 M W_\infty.$$

The proof is found in Appendix A.

III. Bisection Algorithm

In this section, we first provide upper and lower bounds of $\|G\|$. Then we complete a bisection algorithm to compute $\|G\|$ based on Theorem 1 and the bounds.

The bounds of $\|G\|$ given in the following theorem are derived based on the projection used in [5], where bounds for frequency response gain of sampled-data systems are derived.

Theorem 2: $\|G\|$ is bounded from below and above by

$$\gamma_\ell \leq \|G\| \leq \gamma_u$$

where

$$\gamma_\ell := \max(\gamma_{\ell 0}, \sigma_{\max}(D)), \ \gamma_u := \min\left(\gamma_{u0}, \gamma_{\ell 0} + \sqrt{\gamma_{u0}^2 - \gamma_{\ell 0}^2}\right).$$

$\gamma_{\ell 0}$ and γ_{u0} are defined by

$$\begin{aligned} \gamma_{\ell 0} &:= \left(\rho\left(\hat{B}^* \int_0^1 \mathrm{e}^{\hat{A}^* t} \hat{C}^* \hat{C} \mathrm{e}^{\hat{A} t}\, \mathrm{d}t \hat{B}\right)\right)^{1/2} \\ \gamma_{u0} &:= \sigma_{\max}(D) + \sigma_{\max}(Q^{1/2} \Xi^{-1} \Upsilon R^{1/2}) \\ &\quad + \left(\mathrm{trace}\left(C \int_0^1 \int_0^t \mathrm{e}^{As} B B^* \mathrm{e}^{A^* s}\, \mathrm{d}s\, \mathrm{d}t\, C^*\right)\right)^{1/2} \end{aligned}$$

where

$$\begin{aligned} \hat{A} &:= \begin{bmatrix} A & BB^* \\ 0 & -A^* \end{bmatrix} \quad \hat{B} := \begin{bmatrix} -\Xi^{-1} \Upsilon R^{1/2} \\ \mathrm{e}^{A^*} (R^\dagger)^{1/2} \end{bmatrix} \\ \hat{C} &:= [C \quad DB^*] \quad R := \int_0^1 \mathrm{e}^{At} B B^* \mathrm{e}^{A^* t}\, \mathrm{d}t, \end{aligned}$$

and Q is defined in (6). The Moore–Penrose inverse is denoted by $(\cdot)^\dagger$.

The proof is found in Appendix B.

For any given tolerance $0 < \varepsilon < 1$, the following algorithm terminates when γ_U satisfy

$$(1 - \varepsilon)\gamma_U \leq \|G\| \leq \gamma_U. \tag{7}$$

Algorithm 1: Given G and $0 < \varepsilon < 1$.

Initialization: Set γ_L and γ_U by $\gamma_L = \gamma_\ell$, $\gamma_U = \gamma_u$.

While $(\gamma_U / \gamma_L > 1 - \varepsilon)$

Set $\gamma = \sqrt{\gamma_L \gamma_U}$.

Invoke Theorem 1: If $\|G\| < \gamma$, update γ_U by $\gamma_U = \gamma$, else update γ_L by $\gamma_L = \gamma$.

end

Remark 1: The number of the iterations in Algorithm 1, denoted by ν, is determined by

$$\nu - 1 < \log_2 \gamma_u - \log_2 \gamma_\ell - \log_2(1 - \varepsilon) \leq \nu$$

when the initial bound γ_u does not satisfy the stopping criterion (7).

Remark 2: The proposed algorithm is implemented as a part of Sampled-Data Control Toolbox on MATLAB [4].

IV. Special Cases

In this section, we will show reduced versions of Theorem 1 for two special cases. We will also point out that both cases are related to periodic solutions of infinite horizon systems.

Notice formally that $M = 0$ if

$$\Omega \mathrm{e}^{-\mathrm{j}\theta} + \Upsilon = 0. \tag{8}$$

Hence, Theorem 1 is further simplified when (8) holds. Since both Ω and Υ are real matrices, (8) implies either

a) $\Omega = -\Upsilon = I$ ($\theta = 0$);
b) $\Omega = \Upsilon = I$ ($\theta = \pi$).

A. Case of $\Omega = -\Upsilon = I$

In this case, we consider the boundary condition $x(0) = x(1)$. By *gluing* signals related to G, we can study periodic solutions $x \in \mathcal{P}_1$ of the infinite horizon system governed by (1), where $\mathcal{P}_1$ is a set of finite-power periodic signals

$$\mathcal{P}_1 := \left\{ f : f(t) = f(t+1) \quad \int_0^1 f^*(t) f(t)\, \mathrm{d}t < \infty \right\}.$$

In fact the worst case power ratio of the infinite horizon system satisfying $u, y, x \in \mathcal{P}_1$ is equal to $\|G\|$:

$$\sup_{u \in \mathcal{P}_1} \frac{\mathrm{power}(y)}{\mathrm{power}(u)} = \|G\|.$$

The reduced version of Theorem 1 for this case is given as follows.

Corollary 1: Given G with $\Omega = -\Upsilon = I$ and $\gamma > \sigma_{\max}(D) > 0$. Then, the following two statements are equivalent.

i) $\|G\| < \gamma$.
ii) $H_{\min}$ has no eigenvalue on the imaginary axis, or

$$\max_{i \in \{0, 1, \ldots, N\}} \sigma_{\max}(P_i) < \gamma$$

where N is defined as in Theorem 1 with $\theta = 0$.

B. Case of $\Omega = \Upsilon = I$

In this case, we consider the boundary condition $x(0) = -x(1)$. This case is related to finite-power periodic signals f satisfying

$$f(t) = -f(t+1) \quad f(t) = f(t+2).$$

The reduced version of Theorem 1 for this case is given as follows.

Corollary 2: Given G with $\Omega = \Upsilon = I$ and $\gamma > \sigma_{\max}(D) > 0$. Then, the following two statements are equivalent.

i) $\|G\| < \gamma$.
ii) $H_{\min}$ has no eigenvalue on the imaginary axis, or

$$\max_{i \in \{0, 1, \ldots, N\}} \sigma_{\max}(P_i) < \gamma.$$

where N is defined as in Theorem 1 with $\theta = \pi$.

Appendix A
Proof of Theorem 1

The basic procedure to derive Theorem 1 is essentially same with that in [3]: Suppose that we have a unitary operator $U\colon \mathbf{L}_2[0, 1] \to \mathbb{R}^n \oplus X$ for a Hilbert space X satisfying the following two conditions: i) UG^*GU^* is expressed as the sum of a block-diagonal and a finite rank operators

$$\begin{bmatrix} K_0 & 0 \\ 0 & \mathcal{K} \end{bmatrix} + \begin{bmatrix} L_0^* \\ \mathcal{L}^* \end{bmatrix} M_0 \begin{bmatrix} L_0 & \mathcal{L} \end{bmatrix}$$

where $K_0 : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{K} : X \to X$, $M_0 : \mathbb{R}^m \to \mathbb{R}^m$, $L_0 : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{L} : X \to \mathbb{R}^m$. ii) $\gamma^2 I - \mathcal{K} > 0$ holds. Then, $\gamma^2 I - G^*G > 0$ is equivalent to

$$\begin{bmatrix} \gamma^2 I & 0 \\ 0 & \gamma^2 I - \mathcal{K} \end{bmatrix} - \left(\begin{bmatrix} K_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_0^* \\ \mathcal{L}^* \end{bmatrix} M_0 \begin{bmatrix} L_0^* \\ \mathcal{L}^* \end{bmatrix}^* \right) > 0.$$

This turns to

$$\gamma^2 I - \begin{bmatrix} I & 0 \\ 0 & \gamma\mathcal{V} \end{bmatrix} \Phi \begin{bmatrix} I & 0 \\ 0 & \gamma\mathcal{V} \end{bmatrix}^* > 0$$

where

$$\Phi := \begin{bmatrix} K_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_0^* \\ I \end{bmatrix} M_0 \begin{bmatrix} L_0 & I \end{bmatrix}, \mathcal{V} := (\gamma^2 I - \mathcal{K})^{-(1/2)} \mathcal{L}^*.$$

We then have an equivalent condition

$$\rho\left(\Phi \begin{bmatrix} I & 0 \\ 0 & \gamma^2 W \end{bmatrix}\right) < \gamma^2 \quad W := \mathcal{V}^*\mathcal{V} = \mathcal{L}(\gamma^2 I - \mathcal{K})^{-1}\mathcal{L}^*. \tag{9}$$

Noting that $W : \mathbb{R}^m \to \mathbb{R}^m$, (9) is a finite-dimensional condition. In the sequel we derive concrete formulas for K_0, M_0, L_0, and W.

Following [3], let us consider $\Psi : \mathbf{L}_2[0, 1] \to \ell_2$ mapping $f \mapsto \{\varphi_i\}_{i=0}^{\infty}$ defined by

$$\varphi_i := \int_0^1 \mathrm{e}^{-\mathrm{j}\omega_i t} f(t)\, \mathrm{d}t$$

as a candidate of the unitary operator introduced previously. Indeed Ψ is unitary, and the following lemma shows that Ψ satisfies the first condition.

Lemma 1: Assume that $\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^A)$. The (k, ℓ)th block of the matrix expression of $\Psi G^* G \Psi^*$ is given by

$$\delta_{k\ell} P_k^* P_\ell + S_k^* M S_\ell.$$

The proof of Lemma 1 is found in Appendix C.

The second condition also can be verified as follows: It is standard that $\omega \in \mathbb{R}$ satisfies

$$\sigma_{\max}(C(\mathrm{j}\omega I - A)^{-1}B + D) = \gamma$$

if and only if $\mathrm{j}\omega \in \mathrm{eig}(H_{\min})$. Hence, if such an ω exists then

$$\gamma^2 I - P_i^* P_i > 0 \tag{10}$$

holds for $i \geq N+1$. Consequently, we can take

$$\begin{aligned} K_0 &= K \quad \mathcal{K} = \mathrm{diag}(P_{N+1}^* P_{N+1},\ P_{N+2}^* P_{N+2}, \ldots) \\ L_0 &= L \quad \mathcal{L} = \begin{bmatrix} S_{N+1} & S_{N+2} & \cdots \end{bmatrix} \quad M_0 = M. \end{aligned}$$

If no such $\omega \in \mathbb{R}$ exists, (10) holds for any integer i. Hence, we can take K_0 and L_0 as void in the case.

The rest of our task is to give a formula for W which is now defined by

$$W = \sum_{i=i_0}^{\infty} S_i (\gamma^2 I - P_i^* P_i)^{-1} S_i^*$$

where $i_0 := 0$ if $H_{\min}$ has no pure imaginary eigenvalue, otherwise $i_0 := N+1$.

Noting that

$$S_i^* = \tilde{C}(\mathrm{j}\omega_i I - \tilde{A})^{-1}$$
$$(\gamma^2 I - P_i^* P_i)^{-1} = \tilde{D}^{-1}\tilde{C}(\mathrm{j}\omega_i I - H)^{-1}\tilde{B}\tilde{D}^{-1} + \tilde{D}^{-1}$$

we have

$$\begin{aligned}&(\gamma^2 I - P_i^* P_i)^{-1} S_i^*\\ &= \left(\tilde{D}^{-1}\tilde{C}(\mathrm{j}\omega_i I - H)^{-1}\tilde{B}\tilde{D}^{-1} + \tilde{D}^{-1}\right)\tilde{C}(\mathrm{j}\omega_i I - \tilde{A})^{-1}\\ &= \tilde{D}^{-1}\tilde{C}\left((\mathrm{j}\omega_i I - H)^{-1}\tilde{B}\tilde{D}^{-1}\tilde{C} + I\right)(\mathrm{j}\omega_i I - \tilde{A})^{-1}\\ &= \tilde{D}^{-1}\tilde{C}(\mathrm{j}\omega_i I - H)^{-1}(\mathrm{j}\omega_i I - \tilde{A})(\mathrm{j}\omega_i I - \tilde{A})^{-1}\\ &= \tilde{D}^{-1}\tilde{C}(\mathrm{j}\omega_i I - H)^{-1}.\end{aligned}$$

Next, we note that $S_i = J(\mathrm{j}\omega_i I - \tilde{A})^{-1}\tilde{B}$ to have

$$\begin{aligned}&S_i(\gamma^2 I - P_i^* P_i)^{-1} S_i^*\\ &= J(\mathrm{j}\omega_i I - \tilde{A})^{-1}\tilde{B}\tilde{D}^{-1}\tilde{C}(\mathrm{j}\omega_i I - H)^{-1}\\ &= J(\mathrm{j}\omega_i I - \tilde{A})^{-1}\left((\mathrm{j}\omega_i I - \tilde{A}) - (\mathrm{j}\omega_i I - H)\right)(\mathrm{j}\omega_i I - H)^{-1}\\ &= J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right).\end{aligned}$$

Remark 3: This reduction of $S_i(\gamma^2 I - P_i^* P_i)^{-1} S_i^*$ is first pointed out by Mirkin [8].

Invoking [3, Prop. 5], we have

$$\sum_{i=0}^{\infty}(\mathrm{j}\omega_i I - A)^{-1} = \frac{1}{2}(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{A})^{-1}(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A})$$

for θ satisfying $\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^A)$. Hence

$$\begin{aligned}&\sum_{i=0}^{\infty} J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right)\\ &= \frac{1}{2}J(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{H})^{-1}(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{H}) - \frac{1}{2}J(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{\tilde{A}})^{-1}(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{\tilde{A}}).\end{aligned}$$

Noting that

$$\mathrm{e}^{\tilde{A}} = \begin{bmatrix}\mathrm{e}^{-A^*} & -\mathrm{e}^{-A^*}Q\\ 0 & \mathrm{e}^{A}\end{bmatrix}$$

we have

$$(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{\tilde{A}})^{-1} = \begin{bmatrix}\Theta^{-*} & \Theta^{-*}\mathrm{e}^{-\mathrm{j}\theta}Q\Theta^{-1}\\ 0 & \Theta^{-1}\end{bmatrix}\begin{bmatrix}-\mathrm{e}^{A^*}\mathrm{e}^{-\mathrm{j}\theta} & 0\\ 0 & I\end{bmatrix}.$$

where

$$\Theta := \mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{A}.$$

Consequently

$$\begin{aligned}&(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{\tilde{A}})^{-1}(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{\tilde{A}})\\ &= \begin{bmatrix}\Theta^{-*} & \Theta^{-*}\mathrm{e}^{-\mathrm{j}\theta}Q\Theta^{-1}\\ 0 & \Theta^{-1}\end{bmatrix}\begin{bmatrix}-(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A})^* & \mathrm{e}^{-\mathrm{j}\theta}Q\\ 0 & \mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A}\end{bmatrix}\\ &= \begin{bmatrix}-\Theta^{-*}(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A}) & 2\Theta^{-*}Q\\ 0 & \mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A}\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & \Theta^{-1}\end{bmatrix}.\end{aligned}$$

Hence

$$\begin{aligned}&J(\mathrm{e}^{\mathrm{j}\theta}I - \mathrm{e}^{\tilde{A}})^{-1}(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{\tilde{A}})\\ &= \begin{bmatrix}I & 0\\ 0 & \Theta^{-1}\end{bmatrix}^*\begin{bmatrix}0 & -(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A})\\ -(\mathrm{e}^{\mathrm{j}\theta}I + \mathrm{e}^{A}) & 2Q\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & \Theta^{-1}\end{bmatrix}\end{aligned}$$

and, thus, we have

$$\sum_{i=0}^{\infty} J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right) = W_\infty.$$

Then, it is readily that

$$\begin{aligned}&\sum_{i=N+1}^{\infty} J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right)\\ &\qquad = W_\infty - \sum_{i=0}^{N} J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right).\end{aligned}$$

Noting that

$$\begin{aligned}&J(\mathrm{j}\omega_i I - H)^{-1}\\ &= (-\mathrm{j}\omega_i J + HJ)^{-1}\\ &= (-\mathrm{j}\omega_i J + \tilde{A}J - \tilde{B}\tilde{D}^{-1}\tilde{B}^*)^{-1}\\ &= -\left(\begin{bmatrix}C^*C & (\mathrm{j}\omega_i I - A)^*\\ \mathrm{j}\omega_i I - A & 0\end{bmatrix} + \tilde{B}\tilde{D}^{-1}\tilde{B}^*\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}&J(\mathrm{j}\omega_i I - \tilde{A})^{-1}\\ &= (-\mathrm{j}\omega_i J + \tilde{A}J)^{-1} = -\begin{bmatrix}C^*C & (\mathrm{j}\omega_i I - A)^*\\ \mathrm{j}\omega_i I - A & 0\end{bmatrix}^{-1}\\ &= -\begin{bmatrix}0 & (\mathrm{j}\omega_i I - A)^{-1}\\ (\mathrm{j}\omega_i I - A)^{-*} & -(\mathrm{j}\omega_i I - A)^{-*}C^*C(\mathrm{j}\omega_i I - A)^{-1}\end{bmatrix}\end{aligned}$$

we have

$$\sum_{i=0}^{N} J\left((\mathrm{j}\omega_i I - H)^{-1} - (\mathrm{j}\omega_i I - \tilde{A})^{-1}\right) = W_N.$$

This completes the proof of Theorem 1.

Appendix B
Proof of Theorem 2

It is trivial that $\sigma_{\max}(D) \le \|G\|$. We prove other bounds.

Proof of $\gamma_{\ell 0} \le \|G\|$: It is trivial that $\|y\|_2$ for a fixed u satisfying $\|u\|_2 \le 1$ is a lower bound of $\|G\|$. Let us compute $\|y\|_2$ for

$$u(t) = B^*\mathrm{e}^{A^*(1-t)}(R^\dagger)^{1/2}v. \tag{11}$$

where v is a vector of compatible size. Invoking Lemma 2 (a) in Appendix D, we have

$$\begin{aligned}y(t) = &\left(DB^*\mathrm{e}^{A^*(1-t)} + C\int_0^t \mathrm{e}^{A(t-s)}BB^*\mathrm{e}^{A^*(1-s)}\mathrm{d}s\right)(R^\dagger)^{1/2}v\\ &- C\mathrm{e}^{At}\Xi^{-1}\Upsilon R^{1/2}v = \hat{C}\mathrm{e}^{\hat{A}t}\hat{B}v.\end{aligned}$$

Hence

$$\gamma_{\ell 0} = \sup_{\|v\|_2 \le 1}\|y\|_2.$$

Noting that $\|u\|_2 \le \|v\|_2$, we have $\gamma_{\ell 0} \le \|G\|$.

Proof of $\|G\| \le \gamma_{u0}$: Invoking Lemma 2 (a) in Appendix D and the triangle inequality, we have

$$\|G\| \le \|G_1\| + \|G_2\| + \|G_3\|$$

where

$$(G_1 u)(t) := Du(t) \quad (G_2 u)(t) := C\int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,\mathrm{d}s$$

$$(G_3 u)(t) := -C\mathrm{e}^{At}\Xi^{-1}\Upsilon\int_0^1 \mathrm{e}^{A(1-s)}Bu(s)\,\mathrm{d}s.$$

Noting that

$$\|G_2\| \le \|G_2\|_{\mathrm{HS}} = \left(\mathrm{trace}\left(C\int_0^1\int_0^t \mathrm{e}^{As}BB^*\mathrm{e}^{A^*s}\,\mathrm{d}s\mathrm{d}tC^*\right)\right)^{1/2}$$
$$\|G_3\| = \sigma_{\max}(Q^{1/2}\Xi^{-1}\Upsilon R^{1/2})$$

(see, e.g., [2]), we get $\|G\| \le \gamma_{u0}$, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm.

Proof of $\|G\| \le \gamma_{\ell 0} + \sqrt{\gamma_{u0}^2 - \gamma_{\ell 0}^2}$: Define the projection Π related to (11) by

$$(\Pi f)(t) := B^*\mathrm{e}^{A^*(1-t)}R^{\dagger}\int_0^1 \mathrm{e}^{A(1-t)}Bf(t)\,\mathrm{d}t.$$

Note that $\|G\Pi\| = \gamma_{\ell 0}$. Since Π is a projection, we have

$$\|G\|^2 = \gamma_{\ell 0}^2 + \|G(I-\Pi)\|^2.$$
$$\|G(I-\Pi)\|^2 = \|G\|^2 - \gamma_{\ell 0}^2 \le \gamma_{u0}^2 - \gamma_{\ell 0}^2. \tag{12}$$

On the other hand, invoking the triangle inequality, we have

$$\|G\| \le \gamma_{\ell 0} + \|G(I-\Pi)\|.$$

Substituting (12), $\|G\| \le \gamma_{\ell 0} + \sqrt{\gamma_{u0}^2 - \gamma_{\ell 0}^2}$ follows.

This completes the proof of Theorem 2.

Appendix C
Proof of Lemma 1

Denote the output of G for the input

$$u(t) = \mathrm{e}^{\mathrm{j}\omega_i t}I$$

by y_i. Then what we need to compute is $y_k^* y_\ell$. Note that

$$y_i = \mathcal{I}_0(\bar{G}_i)$$

where $\mathcal{I}_0$ is defined in Lemma 3 in Appendix D, and $\bar{G}_i$ is defined by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_i & \bar{B} \\ \bar{C} & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \bar{\Omega}x(0) + \bar{\Upsilon}x(1) = 0$$

where

$$\begin{bmatrix} \bar{A}_i & \bar{B} \\ \bar{C} & 0 \end{bmatrix} := \left[\begin{array}{cc|c} A & B & 0 \\ 0 & \mathrm{j}\omega_i I & I \\ \hline C & D & 0 \end{array}\right], \bar{\Omega} := \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix}, \bar{\Upsilon} := \begin{bmatrix} \Upsilon & 0 \\ 0 & 0 \end{bmatrix}.$$

Then invoke Lemma 3 (a) and Lemma 4 (b) in Appendix D to get

$$y_k^* y_\ell = \mathcal{S}_0\mathcal{I}_0(\bar{G}_k^*\bar{G}_\ell).$$

$\bar{G}_k^*\bar{G}_\ell$ is given by

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{cc|c} -\bar{A}_k^* & \bar{C}^*\bar{C} & 0 \\ 0 & \bar{A}_\ell & \bar{B} \\ \hline -\bar{B}^* & 0 & 0 \end{array}\right]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$
$$\begin{bmatrix} \mathrm{e}^{\bar{A}_k^*}\bar{\Upsilon}^*\bar{\Xi}_k^{-1} & 0 \\ 0 & \bar{\Omega} \end{bmatrix}x(0) + \begin{bmatrix} \bar{\Omega}^*\bar{\Xi}_k^{-1}\mathrm{e}^{\bar{A}_k^*} & 0 \\ 0 & \bar{\Upsilon} \end{bmatrix}x(1) = 0$$

from Lemma 2 (b) and Lemma 4 (a) in Appendix D, where $\bar{\Xi}_i := \bar{\Omega} + \bar{\Upsilon}\mathrm{e}^{\bar{A}_i}$. Then, invoke Lemma 3 (b) in Appendix D to get

$$y_k^* y_\ell = [\,-\bar{B}^* \quad 0\,]\check{\Xi}_{k\ell}^{-1}\begin{bmatrix} \mathrm{e}^{\bar{A}_k^*}\bar{\Upsilon}^*\bar{\Xi}_k^{-1} & 0 \\ 0 & \bar{\Omega} \end{bmatrix}\begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} \tag{13}$$

where

$$\check{\Xi}_{k\ell} := \begin{bmatrix} \mathrm{e}^{\bar{A}_k^*}\bar{\Upsilon}^*\bar{\Xi}_k^{-1} & 0 \\ 0 & \bar{\Omega} \end{bmatrix} + \begin{bmatrix} \bar{\Omega}^*\bar{\Xi}_k^{-1}\mathrm{e}^{\bar{A}_k^*} & 0 \\ 0 & \bar{\Upsilon} \end{bmatrix} \times \exp\left(\begin{bmatrix} -\bar{A}_k^* & \bar{C}^*\bar{C} \\ 0 & \bar{A}_\ell \end{bmatrix}\right).$$

Noting that

$$\exp\left(\begin{bmatrix} -\bar{A}_k^* & \bar{C}^*\bar{C} \\ 0 & \bar{A}_\ell \end{bmatrix}\right) = \begin{bmatrix} \mathrm{e}^{-\bar{A}_k^*} & \mathrm{e}^{-\bar{A}_k^*}\bar{Q}_{k\ell} \\ 0 & \mathrm{e}^{\bar{A}_\ell} \end{bmatrix}$$
$$\bar{Q}_{k\ell} := \int_0^1 \mathrm{e}^{\bar{A}_k^* t}\bar{C}^*\bar{C}\mathrm{e}^{\bar{A}_\ell t}\,\mathrm{d}t$$

we have

$$\check{\Xi}_{k\ell} = \begin{bmatrix} I & \bar{\Omega}^*\bar{\Xi}_k^{-*}\bar{Q}_{k\ell} \\ 0 & \bar{\Xi}_\ell \end{bmatrix}.$$

Hence

$$\check{\Xi}_{k\ell}^{-1} = \begin{bmatrix} I & -\bar{\Omega}^*\bar{\Xi}_k^{-*}\bar{Q}_{k\ell}\bar{\Xi}_\ell^{-1} \\ 0 & \bar{\Xi}_\ell^{-1} \end{bmatrix}.$$

Substituting this into (13), we get

$$\begin{aligned} y_k^* y_\ell &= \bar{B}^*\bar{\Omega}^*\bar{\Xi}_k^{-*}\bar{Q}_{k\ell}\bar{\Xi}_\ell^{-1}\bar{\Omega}\bar{B} \\ &= \bar{B}^*\bar{\Omega}^*\bar{\Xi}_k^{-*}\int_0^1 \mathrm{e}^{\bar{A}_k^* t}\bar{C}^*\bar{C}\mathrm{e}^{\bar{A}_\ell t}\,\mathrm{d}t\,\bar{\Xi}_i^{-1}\bar{\Omega}\bar{B}. \end{aligned} \tag{14}$$

We see that

$$\mathrm{e}^{\bar{A}_i} = \begin{bmatrix} \mathrm{e}^A & (\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^A)(\mathrm{j}\omega_i I - A)^{-1}B \\ 0 & \mathrm{e}^{\mathrm{j}\theta}I \end{bmatrix}$$

to imply

$$\bar{\Xi}_i = \begin{bmatrix} \Xi & \Upsilon(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^A)(\mathrm{j}\omega_i I - A)^{-1}B \\ 0 & I \end{bmatrix}$$

and, hence

$$\bar{\Xi}_i^{-1}\bar{\Omega}\bar{B} = \begin{bmatrix} -\Xi^{-1}\Upsilon(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^A)(\mathrm{j}\omega_i I - A)^{-1}B \\ I \end{bmatrix}.$$

We also have

$$\begin{aligned} \bar{C}\mathrm{e}^{\bar{A}_i t} &= \bar{C}\begin{bmatrix} \mathrm{e}^{At} & (\mathrm{e}^{\mathrm{j}\omega_i t} - \mathrm{e}^{At})(\mathrm{j}\omega_i I - A)^{-1}B \\ 0 & \mathrm{e}^{\mathrm{j}\omega_i t} \end{bmatrix} \\ &= [\,C\mathrm{e}^{At} \quad \mathrm{e}^{\mathrm{j}\omega_i t}I\,]\begin{bmatrix} I & -(\mathrm{j}\omega_i I - A)^{-1}B \\ 0 & P_i \end{bmatrix} \end{aligned}$$

and, hence

$$\bar{C}\mathrm{e}^{\bar{A}_i t}\bar{\Xi}_i^{-1}\bar{\Omega}\bar{B} = [\,C\mathrm{e}^{At} \quad \mathrm{e}^{\mathrm{j}\omega_i t}I\,]\,\Sigma_i$$
$$\Sigma_i := \begin{bmatrix} -\Xi^{-1}(\Omega + \Upsilon\mathrm{e}^{\mathrm{j}\theta})(\mathrm{j}\omega_i I - A)^{-1}B \\ P_i \end{bmatrix}.$$

Substituting this into (14), we get

$$y_k^* y_\ell = \Sigma_k^* \begin{bmatrix} Q & \int_0^1 \mathrm{e}^{A^* t} \mathrm{e}^{\mathrm{j}\omega_\ell t}\,\mathrm{d}t C^* \\ C\int_0^1 \mathrm{e}^{-\mathrm{j}\omega_k t}\mathrm{e}^{At}\,\mathrm{d}t & \delta_{k\ell} I \end{bmatrix} \Sigma_\ell.$$

Noting that

$$C\int_0^1 \mathrm{e}^{-\mathrm{j}\omega_k t}\mathrm{e}^{At}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{j}\theta} C(\mathrm{j}\omega_k I - A)^{-1}(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{A})$$

we finally obtain

$$y_k^* y_\ell = \delta_{k\ell} P_k^* P_\ell + S_k^* M S_\ell.$$

after some manipulations. This completes the proof.

Appendix D
Systems With Two Point Boundary Conditions

The following formulas for manipulating systems with two point boundary conditions are taken from Mirkin and Palmor [9]:

Lemma 2: Let G on $\mathbf{L}_2[0,\,1]$ be given in (1) and (2) with nonsingular Ξ in (3). We have the following.

a) $y = Gu$ is determined by

$$y(t) = Du(t) + \int_0^1 \mathcal{G}(t,\,s)u(s)\,\mathrm{d}s$$

where

$$\mathcal{G}(t,\,s) := \begin{cases} C\mathrm{e}^{At}\Xi^{-1}\Omega\mathrm{e}^{-As}B & (0 \le s < t \le 1) \\ -C\mathrm{e}^{At}\Xi^{-1}\Upsilon\mathrm{e}^{A(1-s)}B & (0 \le t < s \le 1) \end{cases}.$$

b) G^* is given by

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -A^* & C^* \\ -B^* & D^* \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$
$$\mathrm{e}^{A^*}\Upsilon^*\Xi^{-*}x(0) + \Omega^*\Xi^{-*}\mathrm{e}^{A^*}x(1) = 0.$$

Lemma 3: Let G on $\mathbf{L}_2[0,\,1]$ be given in (1) and (2) with nonsingular Ξ in (3). Assume that $D = 0$. Define $\mathcal{I}_0(G) : \mathbb{R}^m \to \mathbf{L}_2[0,\,1]$ by[1]

$$(\mathcal{I}_0(G))(t) := \mathcal{G}(t,\,0). \tag{15}$$

where $\mathcal{G}$ is defined in Lemma 2 in Appendix D. Define also $\mathcal{S}_0 : \mathbf{L}_2[0,\,1] \cap \mathcal{C}[0,\,1] \to \mathbb{R}^p$ by

$$\mathcal{S}_0 y := y(0) \tag{16}$$

where $\mathcal{C}$ denotes the set of continuous functions. Then, we have the following.

a) $(\mathcal{I}_0(G))^* = \mathcal{S}_0 G^*$.

b) Suppose that $CB = 0$. Then, $\mathcal{S}_0\mathcal{I}_0(G) = C\Xi^{-1}\Omega B$.

Lemma 4: Define G_i's $(i = 1, 2)$ on $\mathbf{L}_2[0,\,1]$ by

$$\begin{bmatrix} \dot{x}_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix} \quad \Omega_i x_i(0) + \Upsilon_i x_i(1) = 0$$

[1]The *impulse response* of $\boldsymbol{G}$.

supposing that both Ξ_1 and Ξ_2 are nonsingular, where $\Xi_i := \Omega_i + \Upsilon_i\mathrm{e}^{A_i}$. Assuming the size compatibility, we have the following.

a) $G_1 G_2$ is well-defined and given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \hline y_1(t) \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hline u_2(t) \end{bmatrix}$$
$$\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix}\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = 0.$$

b) $G_1\mathcal{I}_0(G_2) = \mathcal{I}_0(G_1G_2)$.

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